

A K_T -DEFORMATION OF THE RING OF SYMMETRIC FUNCTIONS

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ABSTRACT. The ring of symmetric functions (or a biRees algebra thereof) can be implemented in the homology of $\coprod_{a,b} \mathrm{Gr}_a(\mathbb{C}^{a+b})$, the multiplicative structure being defined from the “direct sum” map. There is a natural circle action (simultaneously on all Grassmannians) under which each direct sum map is equivariant. Upon replacing usual homology by equivariant K-homology, we obtain a 2-parameter deformation of the ring of symmetric functions.

This ring has a module basis given by Schubert classes $[X^\lambda]$. Geometric considerations show that multiplication of Schubert classes has positive coefficients, in an appropriate sense. In this paper we give manifestly positive formulæ for these coefficients: they count numbers of “DS pipe dreams” with prescribed edge labelings.

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Date: March 9, 2015.

The first author was supported by NSF grant DMS-0902296. The second author was supported by Marie Curie International Outgoing Fellowship 2009-254577 of the EU Seventh Framework Program.

1. INTRODUCTION, AND STATEMENT OF RESULTS

1.1. The homology ring. Consider the **direct sum** map on Grassmannians

$$(V, W) \mapsto V \oplus W, \quad \mathrm{Gr}_a(\mathbb{C}^{a+b}) \times \mathrm{Gr}_c(\mathbb{C}^{c+d}) \rightarrow \mathrm{Gr}_{a+c}(\mathbb{C}^{a+b+d+c})$$

inducing the map on homology

$$H_*(\mathrm{Gr}_a(\mathbb{C}^{a+b})) \otimes H_*(\mathrm{Gr}_c(\mathbb{C}^{c+d})) \rightarrow H_*(\mathrm{Gr}_{a+c}(\mathbb{C}^{a+b+d+c})).$$

We can put these maps on homology together to make a trigraded commutative associative ring:

$$\mathcal{R}^H := \bigoplus_{a,b,*} H_*(\mathrm{Gr}_a(\mathbb{C}^{a+b}))$$

The following is well-known, if not usually stated exactly this way:

Theorem. *Let Symm be the (singly-graded) ring of symmetric functions, with basis of Schur functions indexed by partitions. Let $\mathrm{Symm}_{a,b}$ be the subspace linearly spanned by the partitions that fit inside a rectangle of height a and width b . Then the biRees algebra $\bigoplus_{a,b} \mathrm{Symm}_{a,b} s^a t^b \leq \mathrm{Symm}[s, t]$ is isomorphic to \mathcal{R}^H , where the isomorphism takes each Schur function to the Schubert class indexed by the same partition.*

Equivalently, $\mathcal{R}^H / \langle [\mathrm{Gr}_0(\mathbb{C}^{0+1})] - 1, [\mathrm{Gr}_1(\mathbb{C}^{1+0})] - 1 \rangle \cong \mathrm{Symm}$, where the quotient lets one forget the ambient box containing the partition.

In particular, the structure constants in these two rings-with-bases are computable by the same rule, the Littlewood-Richardson rule.

1.2. A two-parameter deformation. Since the direct sum map is equivariant with respect to $\mathrm{GL}(a+b) \times \mathrm{GL}(c+d)$, we might hope to extend the ring to the direct sum of the *equivariant* homologies. For that to work, we'd need one group acting on all the Grassmannians, and the only candidate is $T^\infty \times T^\infty$. Unfortunately, the reindexing¹ of the coordinates involved in the direct sum cuts this down to a single circle action:

Theorem 1.1. *Let S denote a circle acting on each \mathbb{C}^{a+b} with weight 1 on the first b coordinates, weight 0 on the last a , and thereby on $\mathrm{Gr}_a(\mathbb{C}^{a+b})$. Then the direct sum map is S -equivariant, and the induced ring structure on*

$$\mathcal{R}^{HS} := \bigoplus_{a,b,*} H_*^S(\mathrm{Gr}_a(\mathbb{C}^{a+b}))$$

is again commutative associative and trigraded, with a basis (now over $H_S^(\mathrm{pt}) \cong \mathbb{Z}[t]$) given by Schubert classes.*

There is a further deformation available (spoiling the $$ -grading) to the sum*

$$\mathcal{R}^{KS} := \bigoplus_{a,b} K_0^S(\mathrm{Gr}_a(\mathbb{C}^{a+b}))$$

of the S -equivariant K -homology groups, again defining a commutative associative ring (now over $K_S(\mathrm{pt}) \cong \mathbb{Z}[\exp(\pm t)]$).

¹It is actually possible to keep this huge action, at the cost of making the ring be noncommutative and the base ring H_T^* not be central. We did not pursue this, though the geometric techniques used in this paper should generalize to that case.

Since Sym is a polynomial ring (i.e. free), any deformation of its ring structure must be trivializable. Hence there must exist deformations $S_\lambda(t)$ of Schur functions, whose multiplication has the same structure constants as in R^{HS} . We did not succeed in trivializing the family and finding such deformations. Many other deformations of the ring-with-basis of symmetric functions have been studied (e.g. Hall-Littlewood polynomials, Jack polynomials, Macdonald polynomials) and while it is hard to be exhaustive, the two deformations studied here seem to be new.

It is easy to show for geometric reasons that the structure constants of these deformed algebras should be nonnegative in whatever sense appropriate to the cohomology theory by [Kl73] for R^H , [Gr00] for R^{HS} , [Bri02] for R^K , and [AGriMil] for R^{K^S} . The nontrivial contents of the paper are appropriately positive formulæ for these structure constants.

We now go into detail about the bases and coefficient rings. Since coordinates we use in this paper always come in two blocks, the first of size b , the second of size a , we will henceforth be writing $b + a$ rather than $a + b$. In particular, the direct sum map is defined on $\text{Gr}_a(\mathbb{C}^{b+a}) \times \text{Gr}_c(\mathbb{C}^{d+c})$. We will see later why we write its range as $\text{Gr}_{a+c}(\mathbb{C}^{b+d+c+a})$.

If λ is a bit string with content $0^b 1^a$, let M be an $a \times (b + a)$ matrix of rank a whose columns are 0 where λ is 0, and let X^λ denote the **Schubert variety**²

$$X^\lambda := \text{GL}(a) \backslash \overline{\text{GL}(a) M B_{b+a}} \subseteq \text{Gr}_a(\mathbb{C}^{b+a})$$

where B_{b+a} is the upper triangular matrices of size $b + a$, the closure is inside matrices of full rank a , and the identification of X^λ with a subvariety of the Grassmannian is by taking a matrix to its row span.

The dimension of X^λ is the number of inversions (1s occurring somewhere before 0s) in λ . So X^λ is a point when $\lambda = 000 \dots 0111 \dots 1$, in which case M can be taken to be the identity matrix in the *last* a columns.

The Schubert classes $[X^\lambda]$ of partitions $\lambda \subseteq a \times b$ (the box of height a and width b) form bases of homology modules in various homology theories. Four homology theories are of interest to us,

- $H_*(\text{Gr}_a(\mathbb{C}^{a+b}))$ and $K_0(\text{Gr}_a(\mathbb{C}^{a+b}))$, modules over $H_*(\text{pt}) \cong K_*(\text{pt}) \cong \mathbb{Z}$,
- $H_*^S(\text{Gr}_a(\mathbb{C}^{a+b}))$, a module over $H_*^S(\text{pt}) \cong \mathbb{Z}[t]$, and
- $K_0^S(\text{Gr}_a(\mathbb{C}^{a+b}))$, a module over $K_0^S(\text{pt}) \cong \mathbb{Z}[\exp(\pm t)]$.

Letting both λ and their ambient boxes $a \times b$ run, we get bases of the rings R^H , R^{HS} and R^{K^S} as modules over $H_*(\text{pt})$, $\mathbb{Z}[t]$ and $\mathbb{Z}[\exp(\pm t)]$, respectively.

1.3. Pipe dreams for R^{HS} . The datum of a partition $\lambda \subseteq a \times b$ translates into bit string with content $0^b 1^a$ by going from the upper right corner of $a \times b$ to its lower left corner along the boundary of λ and writing down a 0 for each horizontal edge and a 1 for each vertical edge passed along the way. Throughout the article, we move freely back and forth between partitions and bit strings.

Our formulæ for the structure constants of rings-with-bases R^H , R^{HS} and R^{K^S} will be sums over certain square tilings, all edges labeled. The **lower tiles** are the ones shown in figure 1 with a, b in the set of edge **labels** 0, 1, R, Q, and the **upper tiles** are left-right

²If this paper were based on cohomology it would be more natural to call this an *opposite* Schubert variety.

flipped (or upside-down!) versions, with the roles of 0 and 1 flipped. In [Kn] we had many other letters available as labels but we won't need them here.

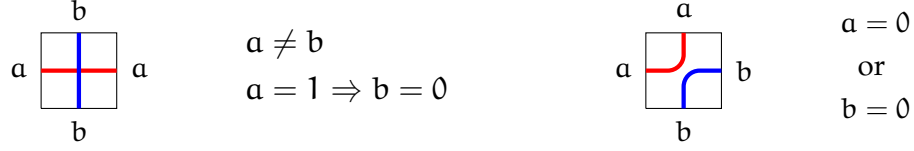


FIGURE 1. Lower tiles, the “crossing” and “elbows”. If $a = b = 0$ in an elbows tile, it is the **equivariant** tile.

Define a **DS pipe dream** (for direct sum³) of partitions $\lambda \subseteq a \times b$ and $\mu \subseteq c \times d$ as a tiling of the region from figure 2, with the following restrictions on the tiles and the labels on the West, South and East boundary:

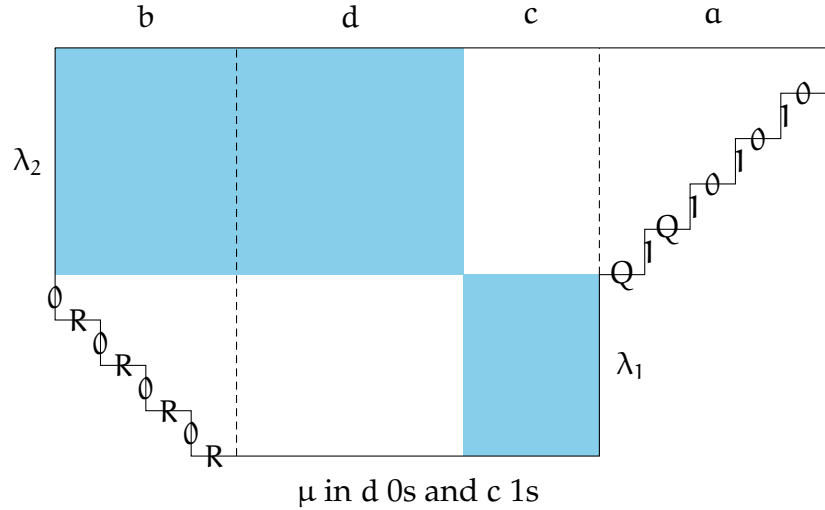


FIGURE 2. The region to be filled by DS pipe dreams

- The tiles in the lower and upper half are lower and upper tiles, respectively. (In fact the boundary conditions ensure that no lower tile used will involve Q.)
- Across the South boundary is the bit string of partition μ read left to right, a string of d 0s and c 1s.
- Write λ as a bit string of b 0s and a 1s. Then *** **recheck** ***
 - λ_1 (on the East side) is the first b letters of λ , read bottom to top, with the 0s turned into Rs and the 1s turned to 0s, and
 - λ_2 is the last a letters of λ , read top to bottom, with the 0s turned into Rs and the 1s turned to Qs

³Pipe dreams come up in seemingly unrelated contexts, such as those in [KnMil04] and [Kn], so it seems safest to include a modifier. The ones here are of closely related to those in [Kn], as the proofs will show.

- There are as many Qs on horizontal edges below the “a” as there are Qs in λ_2 . They are flush left, and the horizontal edges to their right carry 0s. The vertical edges there carry 1s.
- Horizontal edges below the “b” carry Rs, vertical edges carry 0s.
- Any equivariant tiles only appear in the shaded regions.

We will present a more mnemonic picture explaining the labels of the region at the very end of the paper. To a DS pipe dream P of partitions $\lambda \subseteq a \times b$ and $\mu \subseteq c \times d$, we associate a partition $\nu(P)$ depending on only the North labels. The restrictions we impose on the tiles and on the West, South and East labels imply that the only letters appearing on the North labels of the region are 0 and 1. This makes for a word of length $b + d + c + a$ in 0 and 1, which translates into a partition $\nu(P)$.

Define the H^S -**weight** $\text{wt}'_S(P) \in H^S_{\star}(\text{pt}) = \mathbb{Z}[t]$ of a DS pipe dream P as $t^{\#(\text{equivariant tiles})}$. (This will be the leading term, as $t \rightarrow 1$, of the K^S -weight coming later.)

Theorem 1.2. (1) *As elements of $H_{\star}(\text{Gr}_{a+c}(\mathbb{C}^{b+d+c+a}))$, the expansion of $[X^{\lambda}] \cdot [X^{\mu}] = [X^{\lambda} \oplus X^{\mu}]$ in the $H_{\star}(\text{pt})$ -basis of Schubert classes is*

$$[X^{\lambda} \oplus X^{\mu}] = \sum_{\substack{\text{DS-pipe dreams } P \text{ of partitions } \lambda \text{ and } \mu \\ \text{having no equivariant tiles}}} [X^{\nu(P)}] = \sum_{\nu} \# \left\{ \begin{array}{l} \text{DS-pipe dreams } P : \\ \nu(P) = \nu, \\ P \text{ has no equivariant tiles} \end{array} \right\} [X^{\nu}].$$

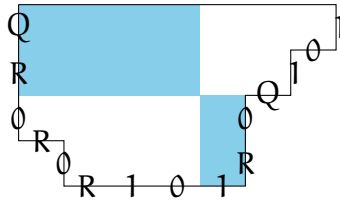
(2) *As elements of $H^S_{\star}(\text{Gr}_{a+c}(\mathbb{C}^{b+d+c+a}))$, the expansion in the $H^S_{\star}(\text{pt})$ -basis is*

$$[X^{\lambda} \oplus X^{\mu}] = \sum_{P \text{ of partitions } \lambda \text{ and } \mu \text{ with no strict } K\text{-tiles}} \text{wt}'_S(P) [X^{\nu(P)}]$$

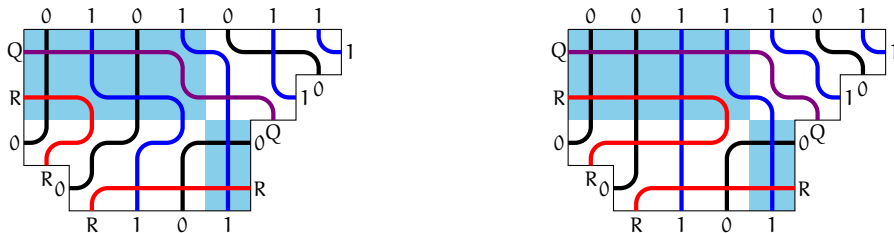
Specializing t to 0 recovers the formula from (1).

The first formula in this theorem is of course just another rule for the standard Littlewood-Richardson number, but we include it to relate it to the H^S -deformation in the second half.

Example 1.3. When expanding the class of $X^{\square} \oplus X^{\square}$ in the basis of Schubert classes, we have to fill the region



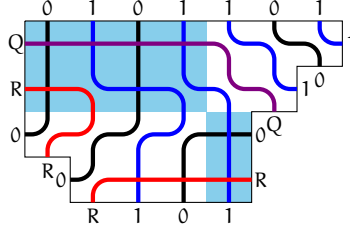
with tiles. We find two pipe dreams using no equivariant tiles,



which says that

$$[X^{\mathfrak{F}} \oplus X^{\mathfrak{P}}] = [X^{\mathfrak{FP}}] + [X^{\mathfrak{P}\mathfrak{F}}]$$

as elements of $H_*(\mathrm{Gr}_4(\mathbb{C}^7))$. We find one pipe dream using an equivariant tile, a 1-1 elbow at $(4, 2)$.

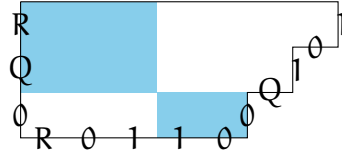


Therefore,

$$[X^{\mathfrak{F}} \oplus X^{\mathfrak{P}}] = [X^{\mathfrak{FP}}] + [X^{\mathfrak{P}\mathfrak{F}}] + t [X^{\mathfrak{FP}\mathfrak{F}}]$$

as elements of $H_*^S(\mathrm{Gr}_4(\mathbb{C}^7))$.

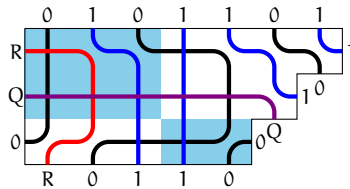
However, since DS pipe dreams are far from symmetric in λ and μ , we also compute the class of $X^{\mathfrak{P}} \oplus X^{\mathfrak{F}}$ to illustrate commutativity. When expanding the class of $X^{\mathfrak{F}} \oplus X^{\mathfrak{P}}$ in the basis of Schubert classes, we have to fill the region



We find two pipe dreams using no equivariant tiles,



and one pipe dream using an equivariant tile, a 1-1 elbow at $(5, 3)$.



The North labels of these pipe dreams are indeed the same as the North labels of the three pipe dreams for $X^{\mathfrak{F}} \oplus X^{\mathfrak{P}}$.

1.4. Pipe dreams for R^{K^S} . For handling the structure constants of the ring-with-basis R^{K^S} , we use more general labels W on the vertical edges of tiles. Each W is a word in $\{1, R, Q\}$ (no 0s), no letters repeating, and if it contains 1 then the 1 must be at the end, so $W \in$

$\{\emptyset, 1, R, Q, RQ, QR, R1, Q1, RQ1, QR1\}$. There are four kinds of **lower K-tiles**, including the fundamentally new “displacer” tile, as shown in figure 3. As in the H^S -case, the crossing tile is subject to the restrictions $W \neq b$ and $W = 1 \Rightarrow b = 0$. The empty word can only appear in the fusor and in the displacer tile; a fusor tile with $W = \emptyset$ is a usual elbow tile as used in DS pipe dreams. If a tile has a word with more than one letter, call it a **strict K-tile**.

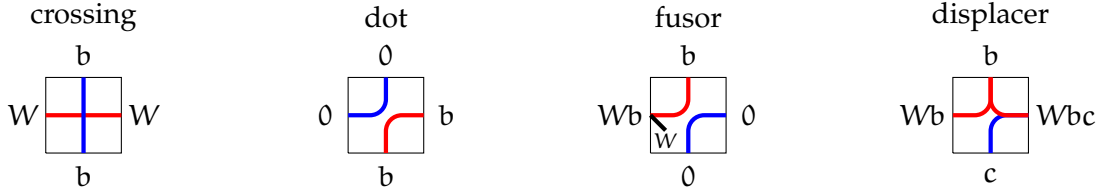
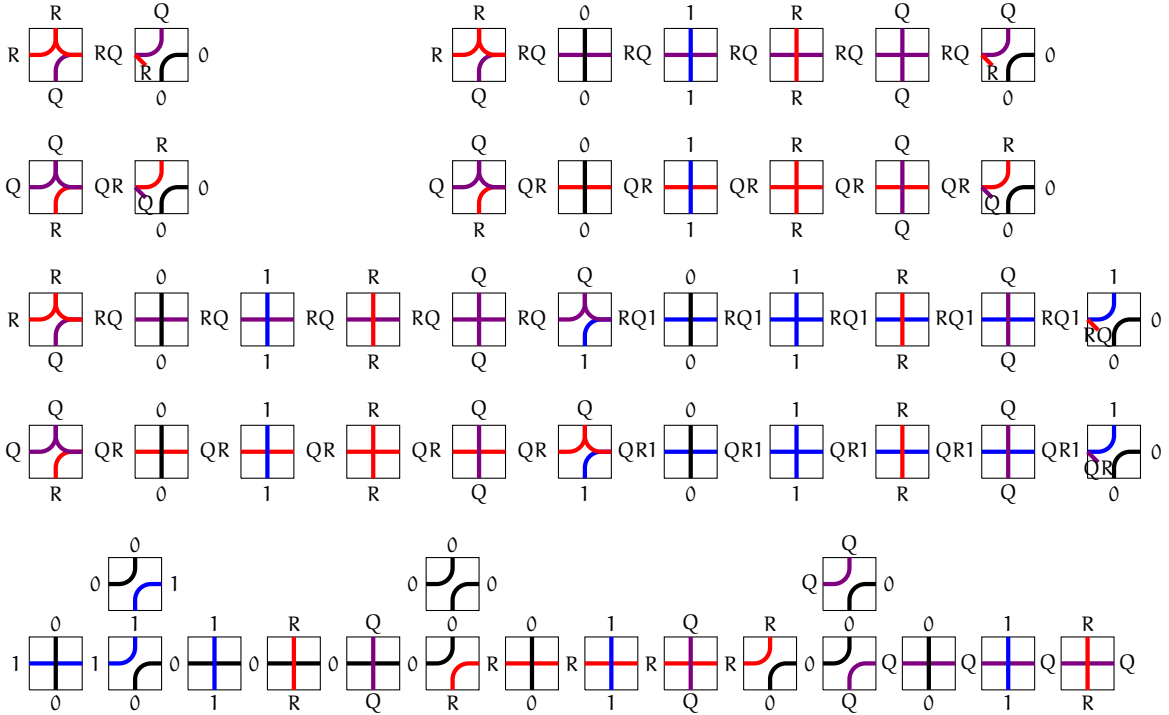


FIGURE 3. Lower K-tiles

There are a total of 53 lower K-tiles. The following list shows all of them in typical positions where alignment happens. The first four lines show honest K-tiles, the rest the tiles for H^S . Only two tiles appear more than once in this list.



The **upper K-tiles** are the left-right (not up-down, any more!) flipped versions of these tiles, with the roles of 0 and 1 exchanged. In particular, no word W properly contains 1, and if it contains 0 then the 0 must be at the end. There are a total of 82 upper and lower K-tiles.

Define a **K-DS pipe dream** of partitions $\lambda \subseteq a \times b$ and $\mu \subseteq c \times d$ as a tiling of the region from the previous subsection, with the same restrictions on the tiles and the boundary

labels as in the previous subsection, but using all types of lower and upper K-tiles, respectively.

Also in this setting, the only letters appearing on the North labels of the region are 0 and 1, which make for a bit string and hence partition $\nu(P)$.

Define the **S-weight** of a K-DS pipe dream P as

$$\text{wt}_S(P) := \prod_{\text{tiles } t} \begin{cases} 1 - \exp(t) & \text{if } t \text{ is the equivariant tile (all 0 labels)} \\ \exp(t) & \text{if } t \text{ is a fusor tile,} \\ 1 & \text{otherwise} \end{cases}$$

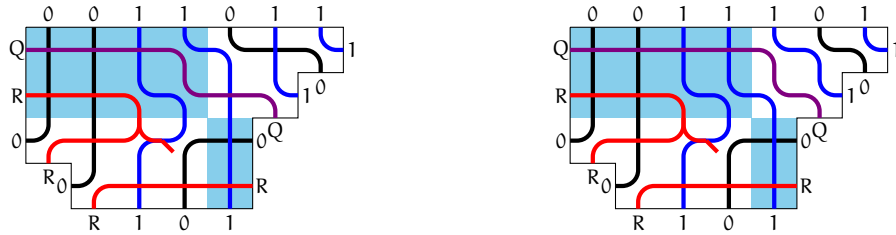
in $K_0^S(\text{pt}) = \mathbb{Z}[\exp(\pm t)]$. (Actually it is natural to define the weight of an equivariant tile in an illegal location to be 0.)

Finally, define $\text{fusing}(P) := |\nu| - |\lambda| - |\mu| + \#\{\text{equivariant tiles}\}$. Equivalently, $\text{fusing}(P)$ is the sum of the sizes of words W appearing in fusor tiles of P . (So $\text{fusing}(P) = 0$ iff the K-DS pipe dream P is an ordinary DS pipe dream, since the presence of a displacer tile forces the appearance of a fusor tile with $|W| > 0$ to the East of it.) We will say more about the two equivalent definitions of $\text{fusing}(P)$ later.

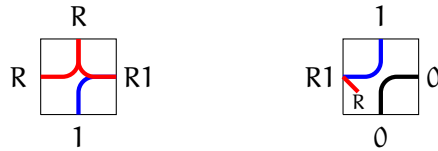
Theorem 1.4. *As elements of $K_0^S(\text{Gr}_{a+c}(\mathbb{C}^{a+b+d+c}))$, the expansion in the $K_0^S(\text{pt})$ -basis is*

$$[X^\lambda \oplus X^\mu] = \sum_{\text{K-DS-pipe dreams } P \text{ of partitions } \lambda \text{ and } \mu} (-1)^{\text{fusing}(P)} \text{wt}_K(P) [X^{\nu(P)}].$$

Example 1.5. When expanding the equivariant K-class of $X^{\overline{\text{ff}}} \oplus X^{\overline{\text{p}}}$ in the basis of Schubert classes, we find the DS pipe dreams from example 1.3, and in addition, two DS pipe dreams containing K-tiles.



Both K-DS pipe dreams have their genuine K-tiles in the lower half, displacer and fusor

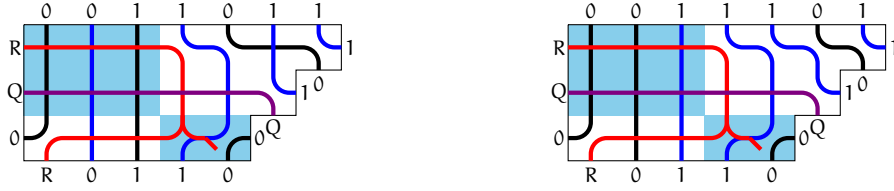


at $(2, 2)$ and $(3, 2)$, with words $W = \emptyset$ and $W = R$, respectively. The second K-DS pipe dream contains an equivariant 1-1 elbow at $(4, 2)$. Therefore,

$$\begin{aligned} [X^{\overline{\text{ff}}} \oplus X^{\overline{\text{p}}}] &= [X^{\overline{\text{ff}}} \overline{\text{ff}}] + [X^{\overline{\text{ff}}} \overline{\text{p}}] + (1 - \exp(t)) [X^{\overline{\text{ff}}} \overline{\text{ff}}] - \exp(t) [X^{\overline{\text{ff}}} \overline{\text{p}}] - (1 - \exp(t)) \exp(t) [X^{\overline{\text{ff}}} \overline{\text{ff}}] \\ &= [X^{\overline{\text{ff}}} \overline{\text{ff}}] + (1 - \exp(t) + \exp(2t)) [X^{\overline{\text{ff}}} \overline{\text{p}}] + (1 - \exp(t)) [X^{\overline{\text{ff}}} \overline{\text{ff}}] - \exp(t) [X^{\overline{\text{ff}}} \overline{\text{p}}]. \end{aligned}$$

as elements of $K_0^S(\text{Gr}_4(\mathbb{C}^7))$.

When expanding the equivariant K-class of $X^{\mathbb{F}} \oplus X^{\mathbb{P}}$ in the basis of Schubert classes, we find the three DS pipe dreams from example 1.3, and in addition, two DS pipe dreams



having the same genuine K-tiles and equivariant tiles.

1.5. Outline of the paper. In §2.1 we recall the definitions of and around positroid varieties, and compute how Grassmannian duality acts on them. In §2.2 we recall the expansion formula from [Kn] that we will twice have to use. The real work is in §3, where we set up the direct sum variety in §3.1, apply the formula the first time in §3.2, use Grassmannian duality in §3.3, and apply the formula the second time in §3.4, with summary in §3.5.

2. INTERVAL POSITROID VARIETIES AND IP PIPE DREAMS

2.1. Positroid varieties, interval positroid varieties, and duality. Our reference for this section is [KnLS].

Each of the expansion formulæ relates a direct sum variety $X^\lambda \oplus X^\mu$ with Schubert varieties $\{X^\nu\}$. To prove them, we make use of a class of varieties that interpolates between them, *interval positroid varieties* (studied in [Kn] and implicitly, in [BiCo12]). While [Kn] gives expansion formulæ for classes of general interval positroid varieties, its equivariant formulae are not positive in the senses required in this paper, as will be explained in §3.2.

First define a **bounded juggling pattern** $J : \mathbb{Z} \rightarrow \mathbb{Z}$ as a bijection satisfying $J(i+n) = J(i) + n$ for all $i \in \mathbb{Z}$ (making it an **affine permutation**), with each $J(i) - i \in [0, n]$. The average value of $J(i) - i$ is necessarily an integer $k \in [0, n]$, called the **ball number**. See Figure 4 for an example; the picture only shows the union of all $\{i\} \times (-i + [0, n])$ in $\mathbb{Z} \times \mathbb{Z}$, which suffices to characterize J .

Given a matrix $M \in M_{k \times n}$, let $(\vec{v}_i)_{i \in \mathbb{Z}}$ be the infinite periodic list of column vectors, $\vec{v}_i :=$ column $i \bmod n$ of M , and associate a bounded juggling pattern J_M defined by

$$J_M(i) := \min \{j \geq i : \vec{v}_i \in \text{span}(\vec{v}_{i+1}, \dots, \vec{v}_j)\}$$

The boundedness and periodicity properties of J_M are obvious, the bijectivity less so but not difficult.

For J a bounded juggling pattern of period n and ball number k , let

$$\Pi_J := \overline{\{\text{rowspan}(M) : M \in M_{k \times n}, J_M = J\}} \subseteq \text{Gr}_k(\mathbb{C}^n)$$

be the associated **positroid variety**. Scheme-theoretically, it is given by the rank conditions

$$\text{rank}(M_{[i,j]} := \text{columns } [i, j] \bmod n \text{ in } M) \leq |[i, j] \setminus J([i, j])|, \quad \forall i \leq j.$$

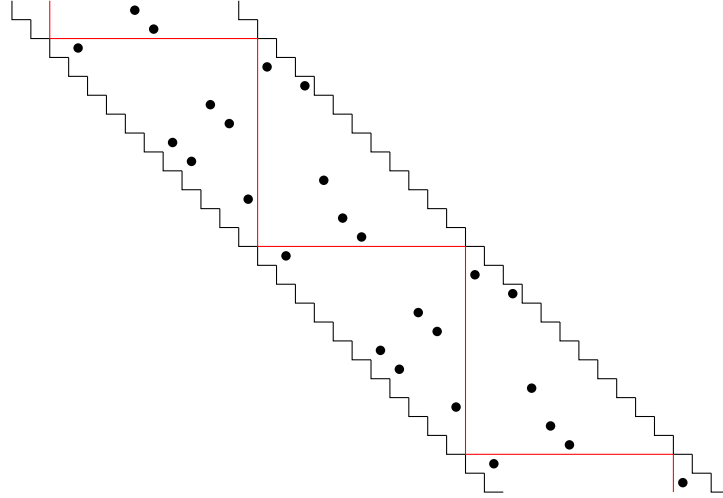


FIGURE 4. A bounded juggling pattern

Some of the rank conditions defining a positroid variety Π_J follow from others. A box (i, j) is called **essential** for J if its rank condition $\text{rank}(M_{[i,j]}) \leq r_{i,j}$ is not implied by the rank condition for any of $(i \pm 1, j)$, $(i, j \pm 1)$. For defining Π_J , it suffices to only impose essential rank conditions $\text{rank}(M_{[i,j]}) \leq r_{i,j}$.

The periodicity condition on J lets us reconstruct it from $J(1), \dots, J(n)$, or equivalently from the parallelogram with top $(1, 1) - (1, n+1)$ and bottom $(n, n) - (n, 2n)$ in the $\infty \times \infty$ permutation matrix. Cut this parallelogram in half along a vertical line into J 's **West triangle** and **East triangle**. In the example of Figure 4, the two triangles are delimited by red lines. By [Kn, §2.2], the positroid variety Π_J is

- an **interval positroid variety**, so called because the only essential conditions are on honest intervals $[i, j] \subseteq [1, n]$ (not cyclic intervals), iff the k dots in the East triangle are Northwest to Southeast,
- a **Richardson variety** iff in addition, the $n - k$ dots in the West triangle are Northwest to Southeast,
- a **Schubert variety**⁴ iff it is a Richardson variety and in addition, the $n - k$ dots in the West triangle are flush North.

Most of the varieties considered in this paper are actually *interval* positroid varieties, which by [Kn, proposition 2.1] can be specified by giving just their West triangles, upper triangular partial permutation matrices of rank $n - k$. The juggling pattern from Figure 4 defines an interval positroid variety, provided the row indices in the West triangle below the red lines are $1, \dots, n$.

This subclass contains the class (depending also on a parameter $i \in [0, n]$) that will actually be of central interest in the paper, and we will call *i-sorted*. If the dots below row i in the West triangle are in the top rows $i + 1, i + 2, \dots$ and run NW/SE, call this being **i-sorted**. Note that we sort from the back, so *i-sorted* implies *j-sorted* when $i \leq j$, not $i \geq j$. Call the West triangle or the bounded juggling pattern it minimally extends to **i-sorted**.

⁴We remind the reader that the convention used here is opposite the cohomological one used in [Kn].

In [Kn], the class of an arbitrary interval positroid variety in T -equivariant K -homology is expressed as a linear combination of Schubert classes. However, due to some reindexing involved in our choice of circle $S \leq T$, that expansion is not positive in the sense required in this paper, so our use of it will be slightly indirect.

Proposition 2.1. *Use the standard bilinear form on \mathbb{C}^n to define a “duality” isomorphism $D : \text{Gr}_k(\mathbb{C}^n) \xrightarrow{\sim} \text{Gr}_{n-k}(\mathbb{C}^n)$. Let $J : \mathbb{Z} \rightarrow \mathbb{Z}$ be a bounded juggling pattern. Then*

$$D(\Pi_J) = \Pi_{J^{-1} \circ (i \mapsto i+n)}.$$

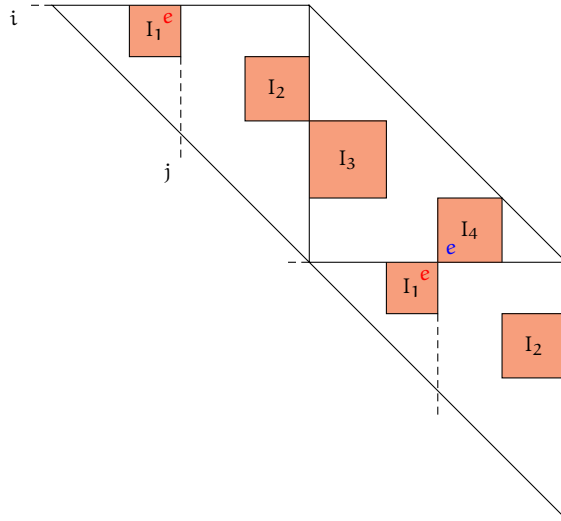
This exchanges the West and East triangles of J , rotating 180° .

Proof. Each Π_J is the intersection of “meet-irreducible” positroid varieties given by single (essential) rank conditions

$$\{M \in M_{k \times n} : \text{rank}(M_{[i,j]}) \leq |[i,j]| - \#\{\text{dots SW of } (i,j) \text{ in } J\}.$$

This says that dots in J come in four blocks of identity matrices:

- Dots in the triangle SW of (i, j) come in an identity block I_1 sitting flush North and flush East in that triangle.
- Dots in the remainder of the West triangle $[i, i+n]$ come in an identity block I_2 sitting due South of I_1 and flush East in that triangle.
- Dots in the complementary East triangle come in the unique NW-SE shape, thus in identity blocks I_3 and I_4 .



The essential condition defining Π_J is indicated by an e . The dual of Π_J is also defined by just one rank condition,

$$\{M \in M_{n-k \times n} : \text{rank}(M_{[j+1, i-1]}) \leq n - k - \text{rank}(I_1)\}.$$

(Perhaps surprisingly, the size of $[i, j]$ does not show up on the right-hand side.) However, we want the rank bound to be $(n - |[i, j]|) - \#\{\text{dots strictly NE of } (i, j)\}$.

Four equalities on the respective sizes of the blocks of J are easy to check,

$$\begin{aligned} \text{rank}(I_1) + \text{rank}(I_2) &= n - k, \\ \text{rank}(I_3) + \text{rank}(I_4) &= k, \\ \text{rank}(I_1) + \text{rank}(I_3) &= [i, j], \\ \text{rank}(I_2) + \text{rank}(I_4) &= n - [i, j]. \end{aligned}$$

By induction over $\text{rank}(I_1)$ (or equivalently, by induction over J in the affine Bruhat order) they imply a fifth equality,

$$\text{rank}(I_4) = \text{rank}(I_1) + k - |[i, j]|$$

giving the desired rank bound. The corresponding essential condition is indicated by the letter e in the picture. \square

2.2. The K^\top -formula from [Kn, §4.4]. We recall several definitions from [Kn], to state (almost) its most precise formula, which involves cutting a triangle $\{(a \leq b)\} \subseteq [n] \times [n]$ into the squares lexicographically before and after a position (i, j) . We simplify very slightly in the translation, in that we only need to cut into the **top half** consisting of rows $[1, i]$ and **bottom half** of rows $[i + 1, n]$.

2.2.1. Slices and their bounded juggling patterns. The basic combinatorial objects in this paper (not the DS pipe dreams, which are visibly composite) are

- (1) $[h, j]$ -*partial pipe dreams*, which for each $i \in [h, j]$ give
- (2) i -*slices*, which give
- (3) i -sorted upper triangular partial permutations, which give
- (4) bounded juggling patterns with East triangle running NW/SE.

We first address (2)→(3)→(4). We have already discussed the last map; [Kn, proposition 2.1] says that there is a unique such extension.

Define an i -**slice** as a labeling with 0, 1, R, Q of the following edges of the top half:

- the $n + 1 - i$ horizontal edges below row i (above row $i + 1$),
- the i vertical edges on the East side of rows $[1, i]$, and
- the $i - 1$ horizontal edges below the diagonal blocks.

(Notice that none of these edges are vertical and interior, which will be why we don't have to consider the more complicated multiple labels that K-pieces can have.)

To an i -slice s , we attempt to associate an upper triangular partial permutation matrix $\pi(s)$, and if successful we call s **viable**.

Draw rays perpendicular to the L-edges (for $L \in \{R, Q\}$), into the top half. There should be the same number m of vertical as horizontal rays – or else s is not viable. They should meet in m^2 locations inside the top half, or else s is not viable. Put dots along the diagonal m of these intersections.

For 1-edges, much the same occurs, except there may be m' horizontal 1s and only $m < m'$ vertical 1s. In this case ignore the left $m' - m$ horizontal 1s and only draw m dots. This completes the filling of the top half.

If there are p 0-labels on the South edge of s , put dots NW/SE in rows $[i + 1, i + p]$ below those 0s. Notice that the first dot will not land inside the bottom half if the label under

the (i, i) square is 0; this is the last way to be non-viable. **Hereafter all slices are assumed to be viable unless stated otherwise.**

The result is richer than an upper triangular partial permutation: its dots come in R-type and Q-type, each group of which separately runs NW/SE. We leave the reader to convince herself that

Lemma 2.2. *This constuction gives a bijection between i -slices, and i -sorted upper triangular partial permutations in which one has chosen a decomposition of the upper dots into the R-dots and Q-dots, each group of which separately runs NW/SE.*

Now that we have filled the triangle with corners $(1, 1), (1, n), (n, n)$ with a partial permutation of dots, there is a unique way to extend it to the permutation matrix of a bounded juggling pattern that is NW/SE in its East triangle [Kn, proposition 2.1]. This we declare to be $g(s)$.

The most important special case is a 0-slice. Since it has no vertical edges, it can have no letters at all, only 0s and 1s. Then (as explained in §2.1, based on [Kn, §2.2]) the resulting $\Pi_{g(s)}$ is just a Schubert⁵ variety. At the other extreme, the $\Pi_{g(s)}$ for n -slices s are Richardson varieties. (With more letters made available, as in [Kn], they are arbitrary interval positroid varieties.)

2.2.2. Partial pipe dreams. Define an (i, j) -**partial pipe dream** P (for $i \leq j$) to be a viable i -slice and j -slice, together with a filling with lower K -tiles of the trapezoid having corners $(i+1, i+1), (i+1, n), (j, n), (j, j)$, compatibly in the sense that any edge is labeled at most once. Note that P has an associated k -slice $s_k(P)$ for each $k \in [i, j]$, but instead of writing $g(s_k(P))$ we'll just write $g_k(P)$.

Theorem 2.3. [Kn, special case of §4.4] *Fix a j -slice s , hence an interval positroid variety $\Pi_{g(s)}$. Let \mathcal{P} be the set of (i, j) -partial pipe dreams P with $s_j(P) = s$. Then as classes in $K^T(\text{Gr}_k(\mathbb{C}^n))$,*

$$[\Pi_{g(s)}] = \sum_{P \in \mathcal{P}} (-1)^{\text{fusing}(P)} \text{wt}_K(P) [\Pi_{g_i(P)}]$$

where

$$\text{wt}_K(P) := \prod_{\text{tiles } t} \begin{cases} 1 - \exp(y_j - y_i) & \text{if } t \text{ is the equivariant tile (it has all 0s) at } (i, j) \\ \exp(y_j - y_i) & \text{if } t \text{ is the lower fusor tile (it appears in the lower half} \\ & \text{and has 0s on its South and East) at } (i, j) \\ 1 & \text{otherwise} \end{cases}$$

in $K_0^T(\text{pt}) = \mathbb{Z}[\exp(\pm y_1), \dots, \exp(\pm y_1)]$, and $\text{fusing}(P)$ is the sum of the sizes of words W appearing in fusor tiles of P .

This inductive theorem was developed for the case $(i, j) = (0, n)$, in which case (in homology) it gives a Littlewood-Richardson rule.

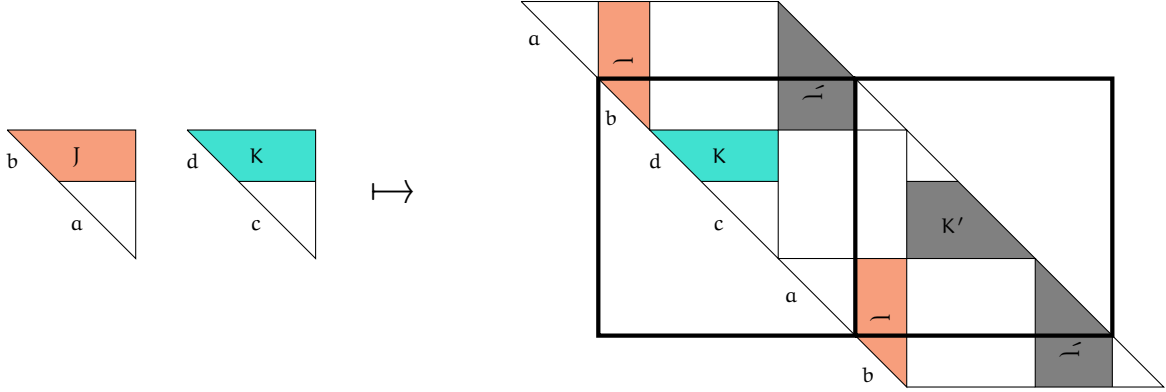
3. DEGENERATION IN STAGES, OF DIRECT SUM VARIETIES

As a way to wrap up the multi-step proof in this section, we provide a road map in §3.5; the reader may want to look ahead to there on a first reading.

⁵in the convention opposite to that of [Kn]

3.1. The positroid variety to start with.

Proposition 3.1. *Let J be the (upper triangular) square matrix from $(1, 1)$ to $(b + a, b + a)$ in the affine permutation matrix whose associated positroid variety is X^λ , and similarly K the $(d + c) \times (d + c)$ one for X^μ . Construct an affine permutation matrix σ as follows: transpose J to $w_0 J^T w_0$ (again upper triangular), direct sum K with that (again upper triangular), extend that as the West half of an affine permutation matrix whose East half runs NW/SE, and (as the bounding squares indicate) rotate forward by a .*



Then Π_σ (which is only a positroid variety, not interval positroid) is K^S -homologous to $X^\lambda \oplus X^\mu$.

In each shaded region (namely J, K, J', K') in this figure, the dots run NW/SE, and only these regions have dots.

Proof. First we consider the σ pictured, then show how to arrive at it geometrically.

Since J and K have dots in all their top rows (b and d dots respectively), when we use $w_0 J^T w_0 \oplus K$ as the West half of an affine permutation matrix, we know that the East half will have no dots East of the North part of K nor North of the East part of $w_0 J^T w_0$. But also, the gray region J' East of the East part of $w_0 J^T w_0$ must have a dots in it (since $w_0 J^T w_0$ misses that many rows out of $a + b$), so has a dot in every column. Hence there are no dots South of J' . Similarly K' (above K) has c dots, and none West of it. We have shown that σ has the form pictured.

On the geometric side, Schubert varieties X^λ and X^μ are defined by rank inequalities on points $\text{rowspan} \begin{bmatrix} B & A \end{bmatrix} \in \text{Gr}_a(\mathbb{C}^{b+a})$ and $\text{rowspan} \begin{bmatrix} D & C \end{bmatrix} \in \text{Gr}_c(\mathbb{C}^{d+c})$ where B, A, D and C are block matrices of sizes $b \times a, a \times a, d \times c$ and $c \times c$, respectively. Since the weight of the S -action on $\text{Gr}_c(\mathbb{C}^{d+c})$ is 1 on block B and 0 on block A , flipping the numbering of columns within blocks B and A is an S -equivariant automorphism

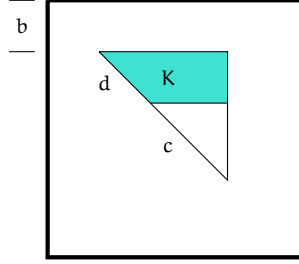
$$\text{rowspan} \begin{bmatrix} B & A \end{bmatrix} \mapsto \text{rowspan} \begin{bmatrix} \overleftarrow{B} & \overleftarrow{A} \end{bmatrix}$$

of $\text{Gr}_a(\mathbb{C}^{b+a})$. We denote by $\overleftarrow{X}^\lambda$ the image of X^λ under this automorphism. Essential conditions on points in X^λ only involve initial intervals columns starting in the first column. Essential conditions on their flipped counterparts therefore involve periodic intervals going from column b to the left, then wrapping around the matrix and going from the last column to the left.

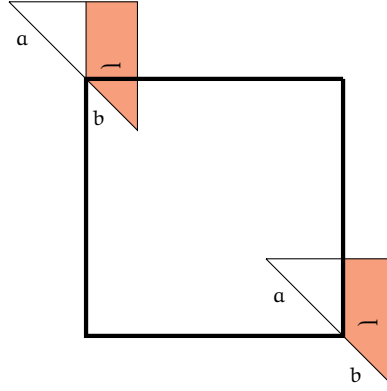
We obtain the variety $\overleftarrow{X}^\lambda \oplus X^\mu$ by imposing the rank inequalities defining $\overleftarrow{X}^\lambda$ and X^μ , respectively, on the columns of matrix

$$\left[\begin{array}{c|c|c|c} \overleftarrow{B} & 0 & 0 & \overleftarrow{A} \\ \hline 0 & D & C & 0 \end{array} \right].$$

Essential conditions on the blocks $\left[\begin{array}{c|c} D & C \end{array} \right]$ of the matrix are implemented by the block



of σ . They only concern honest intervals ranging in columns $b + 1, \dots, b + d + c$ in the matrix. Essential conditions on the complementary blocks of the matrix also involve periodic intervals. They are implemented by the block



of σ . Therefore,

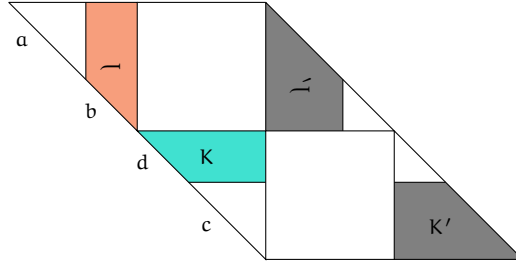
$$\Pi_\sigma = \overleftarrow{X}^\lambda \oplus X^\mu$$

is K^S -homologous to $X^\lambda \oplus X^\mu$, and a positroid variety, not interval positroid. \square

Though it is not obvious why yet, it will turn out also to be useful to backward-rotate by b , thus working with matrices

$$\left[\begin{array}{c|c|c|c} \overleftarrow{A} & \overleftarrow{B} & 0 & 0 \\ \hline 0 & 0 & D & C \end{array} \right].$$

defining points in the Richardson variety associated to the affine permutation matrix



which we call σ' . Of course this only gives the same class in $K^S(\text{Gr}_{a+c}(\mathbb{C}^{b+d+c+a}))$ if we also change the S -action, now to be weight 0 on the first a and last c coordinates, and weight 1 on the $b + d$ coordinates in the middle.

3.2. The bottom $b + d + c$ rows, of which $d + c$ are trivial. Our goal now is to calculate the class $[\Pi_{\sigma'}] \in K^S(\text{Gr}_{a+c}(\mathbb{C}^{a+b+d+c}))$, where S acts with weight 0 on the first b and last d coordinates, and weight 1 on the $a + c$ coordinates in the middle. If the weights were monotonic, we could use theorem 2.3 directly; since they instead go $0^a 1^{b+d} 0^c$ it will take three steps, this being the first. To emphasize: this nonmonotonicity is the reason that this paper is not a straightforward corollary of [Kn].

Theorem 3.2. *Let σ' be constructed from J and K as pictured above, and S act on $\mathbb{C}^{a+b+d+c}$ with weight 1 on the b and d groups of coordinates. Then σ' is $(a + b)$ -sorted, so there exists a unique $(a + b)$ -slice s with no Q -labels such that $g(s) = \sigma'$. Let $\mathcal{P} := \{\text{the } (a + 1, a + b)\text{-partial pipe dreams } P \text{ with } s_{a+b}(P) = s\}$ (which will also have no Q s). Then*

$$[\Pi_{\sigma'}] = \sum_{P \in \mathcal{P}} \text{wt}_S(P) [\Pi_{g(s_a(P))}]$$

as classes in K^S , where we insist too that any equivariant tiles only occur in the rightmost c columns.

Proof. The $(a + b)$ -sortedness is manifest, and we apply lemma 2.2 to obtain s , then apply theorem 2.3 to get the sum.

However, that theorem gives the T^n -equivariant K -class of $\Pi_{\sigma'}$, not the coarser S -equivariant class. To get the K^S -class, we specialize the y_i for the b and d groups of coordinates to t , and those for the a and c to 0. That kills any summand with an equivariant piece outside the rightmost c columns, and otherwise specializes $\text{wt}_K(P)$ to $\text{wt}_S(P)$. \square

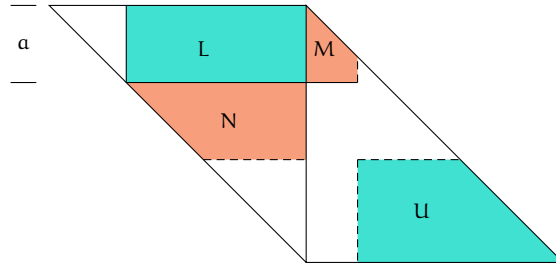
We could have applied theorem 2.3 all the way to $i = 0$, rather than filling in tiles only below row $i = a$, but the specialization of that formula from K^T to K^S would not be positive in the sense of [AGriMil], because of this nonmonotonicity.

These $(a + 1, a + b)$ -partial pipe dreams from this theorem will be the lower halves of the DS pipe dreams appearing in our main theorem 1.4.

3.3. Duality at mid-sort. Before going to $i = 0$ in theorem 2.3 we make use of Grassmannian duality.

Proposition 3.3. *Consider the $(a + 1, a + b)$ -partial pipe dreams \mathcal{P} from theorem 3.2. For each $P \in \mathcal{P}$, not only is $\Pi_{g(s_a(P))}$ interval positroid, but its rotation by a is dual interval positroid, and that dual is a -sorted.*

Proof. First we look at $s_a(P)$, whose labels below the squares $(1, 1) \dots (a, a)$ agree with those on $s_{a+b}(\sigma')$, and hence are all 0s. We claim $g(s_a(P))$ has the form

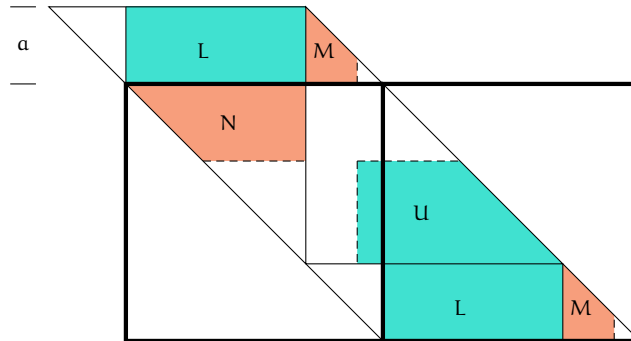


(where L and M have height a and the dashed lines are to indicate that the heights of N and U and the widths of M and U are not determined by a, b, c, d), with dots NW/SE in each of the shaded regions L, M, N, U (and only there). Why?

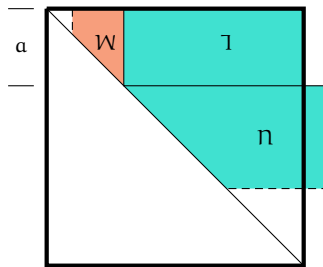
- $g(\text{an } a\text{-slice})$ is a -sorted, so N is NW/SE and there are no dots East of N .
- The a -slice only has one kind of letter, R , so the dots in L must run NW/SE.
- The East half is NW/SE by the definition of $g()$. Hence if M is the region above N 's rows, and U the region below, then M must be West of U and M, U must each run NW/SE.

Moreover, every row of N and U have dots, since those are the only places where the dots could be in those rows.

Now consider the window on $g(s_a(P))$ rotated forward by a :



The dots in the West square run NW/SE, which says that the *dual* (obtained by rotating the two-square domino by 180° , as explained in proposition 2.1) is an interval positroid variety. Moreover, since U runs NW/SE with a dot in every row, when we rotate (to dualize) that East square to σ'' ,



we get something a -sorted. □

The duality isomorphism $\text{Gr}_{a+c}(\mathbb{C}^{b+d+c+a}) \cong \text{Gr}_{b+d}(\mathbb{C}^{b+d+c+a})$ is S -equivariant if in the latter coordinates, S acts now with weight -1 on the b and d groups of coordinates (and 0 on the others). It remains to compute the classes of these $\{\Pi_{s''}\}$ in $K^S(\text{Gr}_{b+d}(\mathbb{C}^{b+d+c+a}))$.

3.4. The final a rows.

Proposition 3.4. *Fix an $(a+1, a+b)$ -partial pipe dream $P \in \mathcal{P}$ from theorem 3.2, giving the rotated dual σ'' from the end of proposition 3.3. Split the dots in its upper half into Q -dots in M , R -dots in L , and let s be the corresponding a -slice (as in lemma 2.2).*

Let \mathcal{P}' be the set of $(0, a)$ -partial pipe dreams P'' such that $s_a(P'') = s$, where we use 180° rotations of the lower K -tiles and we insist that any equivariant tiles only occur in the rightmost $b+d$ columns.

If we flip the rows of $P'' \in \mathcal{P}'$ left-right while trading 0 s for 1 s, then the labels below the a th row of P'' match the labels below the a th row of the original P .

Proof. Rather than flipping each time, it will be easier to rotate σ'' to better match the East half of the second figure in proposition 3.3. So for short, let τ be this 180° rotation of $s_a(\sigma'')$. We also rotate the partial permutation $\pi(s_a(\sigma''))$ by 180° , thus obtaining a partial permutation θ living in the lower triangle of the ambient square of τ .



We have to inspect the duality operation to see how the labels relate. For each column $C \leq b+d+c$ of the rotated slice τ , there is either

- a dot in the last a rows of θ (the L part), in which case the label in column C of τ is R ,
- a dot higher up (the U part), in which case the label in column C of σ'' is 0 ,
- no dot in C , in which case the label in column C of σ'' is 1 .

Correspondingly, in column $a+C$ of σ , we have

- a dot in row $\leq a$ (the L part), in which case the label in column $a+C$ of $s_a(P)$ is R ,
- no dot in column $a+C$ (since thinking of U above N , they can't both have a dot), in which case the label in column $a+C$ of $s_a(P)$ is 1 ,
- a dot below row a (the N part), in which case the label in column $a+C$ of $s_a(P)$ is 0 .

Now consider the last a columns of τ (above M in the figure), i.e. on the diagonal. These are all Q s, then all 1 s. The number of Q s is the number of dots in M , This is also the number of empty rows of L , each of which comes from a 0 or 1 in the left $b+d+c$ columns of τ . The labels on the left edge of τ are R s and Q s the former sitting in the same rows as L 's dots, the latter in the same rows as M 's.

- Each dot in K sees a 0 on the horizontal edge North of it. All other horizontal labels there are 1s, hence the word μ .
- Each dot in the lower half of J sees an R on the vertical edge East of it. All other labels there are 0s, hence the word λ_1 .
- Each dot in the upper half of J sees an R on the vertical edge West of it. Each dot in J' sees a Q on the vertical edge West of it. Every row in the upper half contains a dot, hence the word λ_2 .
- Each dot in J sees an R on the horizontal edge South of it. Since every column in J contains a dot, there are only horizontal Rs South of J .
- The labels on vertical edges South of J are all 0s.

- Each dot in J' sees a Q on the horizontal edge South of it. Since only the first few columns on the West of J' contain dots, this makes for a few Q s, followed by 0 s, on horizontal edges South of J .
- The labels on vertical edges South of J' are all 1 s.

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